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Note

Double Precision FORTRAN Subroutines to Compute both Ordinary and Modified Bessel Functions of the First Kind and of Integer Order with Arbitrary Complex Argument: $J_n(x + jy)$ and $I_n(x + jy)$*

Both ordinary and modified Bessel functions of the first kind and of integer order and complex argument arise in many areas of mathematical physics. For example, in the field of acoustics the complex Bessel functions $J_0(x + jy)$ and $J_1(x + jy)$ occur in the dispersion relation resulting from the steady-periodic wave propagation in a compressible viscous liquid contained in a rigid impermeable tube waveguide [1].

Because of the potentially large numbers involved for complex argument of arbitrary modulus and phase, the very rapid oscillation that occurs in various regions of the complex plane, and the restrictive asymptotic series that unnecessarily limits the large argument solution space, current computer programs restrict the argument of J_n and I_n to either pure real or complex with small modulus [2]. Thus, it is desirable to relax these limitations and to extend the argument into the entire complex domain. This note describes two FORTRAN subroutines for calculating in double precision the ordinary and modified Bessel functions of the first kind and integer order for any point in the complex plane.

Since $J_n(x + jy)$ of large modulus $|x + jy|$ cannot be computed directly under certain asymptotic conditions [3, p. 364, Eq. (9.2.5)], it is best to write it in terms of $I_n(x + jy)$, which has no such problem.

The ascending series for $I_\nu(z)$ with ν and $z = x + jy$ complex as given by Abramowitz and Stegun [3, p. 375, Eq. (9.6.10)] is easily written in the recursive form,

$$I_n(z) = \sum_{k=0}^{\infty} T_k, \qquad T_k = \frac{C}{k(k+n)} T_{k-1}, \qquad T_0 = \frac{C_1}{n!}, \qquad \text{(1a, b, c)}$$

with

$$C = \frac{z^2}{4}, \qquad C_1 = \left(\frac{z}{2}\right)^n, \qquad \text{(2a, b)}$$

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where ν has been replaced by integer n. Degenerate properties of $I_n(z)$ that are needed in the calculation include: (i) $I_n(z)$ is indeterminate at $z = 0$ [3, p. 375, Eq. (9.6.7)] and possesses the values $I_0(0) = 1$ and $I_{|n|>0}(0) = 0$; (ii) $I_n(z)$ exists for n a negative integer and is given by the relation [3, p. 375, Eq. (9.6.6)] $I_{-n}(z) = I_n(z)$; (iii) $I_n(z)$ is pure real for either $z = x$ or ($z = jy$ and n even); and (iv) $I_n(z)$ is pure imaginary for $z = jy$ and n odd.

$I_n(z)$ is an entire function of z and, hence, possesses an infinite radius of convergence. However, for the finite amount of significance available on a digital computer, truncation and round-off error effectively reduce the radius of convergence to a finite value for a prescribed accuracy criterion. Thus, to compute the modified Bessel function for a modulus larger than the ascending series effective radius of convergence $R_{\max}$, an asymptotic series expansion for $I_n(z)$ is required.

The commonly referred to asymptotic series for $I_\nu(z)$ [3, p. 377, Eq. (9.7.1)] excludes the case of $|\arg(z)| = \pi/2$, and thus cannot be used. Fortunately, $I_\nu(z)$ can be written in terms of the more general confluent hypergeometric function of the Kummer type. Transforming $I_\nu(z)$ to this form and coupling it with analytic continuation will then allow the complete specification of $I_\nu(z)$ for all large moduli. $I_\nu(z)$ written in terms of the Kummer function of the first kind is given as [3, p. 377, Eq. (9.6.47)]

$$I_n(z) = \left(\frac{z}{2}\right)^n \frac{e^{-z}}{n!} M(n + \tfrac{1}{2}, 2n + 1, 2z), \tag{3}$$

where ν has been replaced by integer n.

Substituting the desired asymptotic expansion for the Kummer function of the first kind [3, p. 508, Eq. (13.15.1)] into Eq. (3), and again writing the result in recursive form, yields the relation

$$I_n(z) = \frac{(2n)!}{2^{2n+1/2} z^{1/2} n!\, \Gamma(n + \frac{1}{2})} \left[e^{j\pi(n+1/2)-z} \left\{ \sum_{\nu=0}^{R-1} T_\nu(-) + 0\left(\frac{1}{|2z|^R}\right) \right\} \right.$$

$$\left. + e^z \left\{ \sum_{\nu=0}^{S-1} T_\nu(+) + 0\left(\frac{1}{|2z|^S}\right) \right\} \right] \qquad (0 \leqslant \arg(z) \leqslant \pi),$$

$$T_\nu(\pm) = T_{\nu-1}(\pm) \frac{(n + \nu - \frac{1}{2})(-n + \nu - \frac{1}{2})}{\nu} C(\pm), \qquad T_0(\pm) = 1, \tag{4a, b, c}$$

with [3, p. 255, Eq. (6.1.12); p. 3]

$$C(\pm) = \pm \frac{1}{2z}, \qquad \Gamma\left(n + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}{2^n} \Gamma\left(\frac{1}{2}\right),$$

$$\Gamma(\tfrac{1}{2}) = 1.772453850905516027. \tag{5a, b, c}$$

Equation (4a) actually is valid in quadrants III and IV, excepting on the negative imaginary axis. Because analytic continuation would be required in either case, the above overrestriction has been introduced for simplicity. For any z in the region $(-\pi < \arg(z) < 0)$, use of analytic continuation allows $I_n(z)$ to be computed in quadrants I and II by the relation [3, p. 376, Eq. (9.6.30)],

$$I_n(z) = e^{-n\pi j} I_n(ze^{\pi j}) \qquad (-\pi < \arg(z) < 0). \tag{6}$$

Having computed $I_n(z)$, it is now easy to compute $J_n(z)$ from the relation [3, p. 375, Eq. (9.6.3)],

$$J_n(z) = \begin{cases} e^{n\pi j/2} I_n(ze^{-\pi j/2}), & \text{if } \ (y \geqslant 0) \text{ or } (x > 0 \text{ and } y < 0), \\ e^{-3n\pi j/2} I_n(ze^{3\pi j/2}), & \text{if } \ x \leqslant 0 \text{ and } y < 0. \end{cases} \tag{7}$$

The above relations for $J_n(z)$ and $I_n(z)$ have been programmed for a Univac 1108 computer using FORTRAN 5. The Univac 1108 carries single precision complex numbers to nine place significance and magnitude $10^{\pm 38}$ and double precision real numbers to eighteen place significance and magnitude $10^{\pm 308}$. After attempting to compute $I_n(z)$ in single precision complex arithmetic, it was found that not enough accuracy could be obtained due to truncation and round-off error. Also, the magnitude of the numbers involved tended to be of order $10^{\pm 50}$, and so beyond the scope of the single precision arithmetic. As no double precision complex arithmetic exists for the Univac 1108 software, these numerical difficulties forced the writing (in FORTRAN) of a double precision complex arithmetic.

Three methods were employed to test the accuracy of the above routines. In the first method, the series for $I_n(z)$ was substituted into Bessels equation $LI_n(z) = \epsilon$, where $L = z(d/dz)(z(d/dz)) - z^2 - n^2$ and ϵ is an absolute error. ϵ would be zero if $I_n(z)$ were completely accurate. The second method involved checking the values of $I_n(z)$ against known tabular data [3, Table 9; 4]. In the third and most crucial method, the values of $I_n(z)$ obtained from the ascending series, Eq. (1), were compared to those obtained from the asymptotic series, Eq. (4), in the annular region where both series are valid. The range of parameters n and $z = Re^{j\theta}$ considered in the test program was $(n = 0(10)1)$, $(R = 0(100)2)$, and $(\theta = 0(2\pi)\ \pi/12)$.

Defining ϵ_r to be the modulus of the ratio of the N-th term to the N-th partial sum, and relative error δ_r to be the ratio $|LI_n(z)/I_n(z)|$, the results of the three tests on the ascending series show, for ϵ_r of 10^{-15}, a maximum accuracy δ_r of 10^{-16} for modulus R of 6 or less. The relative error then increased steadily from $\delta_r = 10^{-10}$ at $R = 16$, $\delta_r = 10^{-8}$ at $R = 20$, $\delta_r = 10^{-4}$ at $R = 30$, until finally, $R = R_{\max} = 40$, $\delta_r = 1$. The last of these results implies that, due to truncation and round-off, the error is of the same order of magnitude as the value of the function at this large radius. Thus, for $|z| > 40$, the results of the ascending series are rendered meaningless.

Turning next to the asymptotic series, trial calculation showed that, for $\epsilon_r = 10^{-15}$, a minimum stable radius $R_{\min} = 16.5$ produced a minimum relative error $\delta_r = 10^{-11}$. This error stayed relatively constant over the entire range tested. Because of the essential singularity $(1/z^n)$, which causes the asymptotic series to diverge if a sufficiently large number of terms are included, δ_r decreased drastically for R less than 16.5.

Comparison of the two series in the overlapping annular region ($R_{\min} \leqslant R \leqslant R_{\max}$) revealed that the minimum stable error was produced for $n = 0$ when the ascending series were used for moduli less than 18 and the asymptotic series were used for moduli greater than 18. This boundary radius for higher n was found to vary linearly with n according to the relation

$$R_m = 18 + \tfrac{3}{7}n \qquad (0 \leqslant n \leqslant 10). \tag{8}$$

No values of n greater than 10 were tried. Use of Eq. (8) resulted in a minimum of 9 decimal place accuracy for the ascending series and 11 decimal place accuracy for the asymptotic series.

The Bessel functions $J_n(z)$ and $I_n(z)$ were compiled as the subroutines JNUZ(N, X, Y, E, F) and INUZ(N, X, Y, E, F), respectively. The above arguments are interpreted as $E + jF = J_N(X + jY)$ and $E + jF = I_N(X + jY)$, respectively. Arguments N, X, Y, E, and F are all real double precision numbers. The actual program listings for JNUZ, INUZ, and the double precision complex arithmetic appear in the dissertation by Scarton [1, pp. 657–664]. Also appearing in this work is additional information concerning the calculation method [1, pp. 595–625].

As a final note, it should be pointed out that JNUZ and INUZ can be easily modified to include complex N.

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